|  |
| --- |
| **Question Pack – Math Probability** |
| [Type the document subtitle] |
|  |
|  |
|  |
|  |
| **9/23/2007** |
|  |

[Permutation 3](#_Toc186214061)

[Combination 3](#_Toc186214062)

[MultiChoose & Multiset 3](#_Toc186214063)

[Multiset Permutation 4](#_Toc186214064)

# Definitions

## Sets and Multiset

Multiset is more generalized form of a set. An element can occur in the set more than once. Number of times an element occurs is referred by multiplicity of that number. In a multiset {a, a, b} is not the same as {a, b}. {a, a, b} is a multiset whose cardinality is 3. In multi set order of the elements is not important so {a, b, c, d} is the same as {b, c, d, a}.

# Permutation (Order matters)

## Without repetition

Number of ways to arrange k elements from a set of size n where order is important is

$$P\_{k}^{n}= n x (n-1) x (n-2) x … (n – k + 1)=\frac{n!}{\left(n-k\right)!}$$

This is because the first element can be selected in n ways, second element in n-1 ways, third n-2 ways and so on. Number of ways to rearrange all n elements is n!

## WITHOUT REPETITION and WHEN m element are always present

Number of ways to arrange n elements in k spots, such that m elements are always present.

Number of ways *m* elements can be placed in k places = $P\_{m}^{k}$

Number of ways the remaining *k*-*m* places can be filled with n-m elements = $P\_{k-m}^{n-m}$

Total number of ways $=P\_{m}^{k}×P\_{k-m}^{n-m}$

## With repetition

When repetitions are involved, each element can be selected in n ways. Total number of ways k elements be selected

$$Number of ways=n×n×n…= n^{k}$$

## WITH REPETITION Multiset

The letters “ABC” can be arranged in 6 different ways. They are ABC, ACB, BAC, BCA, CAB, CBA. The first position can be chosen in 3 ways, second position can be selected in 2 ways (because one letter is already selected in position 1) and third one in 1 way. Hence the total number of permutations can be expressed as 3 X 2 X 1 or 3!.

If there are n distinct letters, number of ways they can be arranged is given by n!

If some letters are repeated as in SEED. You need to first find total permutations as if they all are distinct. So it is 4!. Since there are 2 instances of E, to identify each instance let us refer it as E1 and E2. SE1E2D and SE2E1D are the same. (i.e.) # of ways EE can be shuffled is eliminated. This reduces the permutations by 2!. # of ways is given by 4!/2!. Likewise if E is duplicated 3 times, it reduces the permutations by 3!

In the example of SEED, is a simple multi set with 3 distinct elements SED, with multiplicity of 1,2 and 1 respectively. For a generic multi set of n elements, with r distinct elements with respective multiplicity n1,n2, n3…nr is given by

$$\left(\genfrac{}{}{0pt}{}{n}{n\_{1}n\_{2}..n\_{r}}\right)=\frac{n!}{n\_{1}!n\_{2}!..n\_{r}!}, Where n=n\_{1}+ n\_{2}+ … n\_{r}$$

This is called multinomial coefficient. As a special case when r=2 is called Binomial Coefficient; Hence n = n1 + n2. If n1 = k, n2 = n – k. Hence

$$\left(\genfrac{}{}{0pt}{}{n}{k}\right)=\frac{n!}{k!\left(n-k\right)!} $$

## CIRCULAR PERMUTATION

The elements are arranged in a circle form. In case of linear permutations, ABCD, BCDA, CDAB, DABC are different, while in case of circular permutation they are the same.

To find the number of distinct circular permutations is by fixing one element and varying the rest. Which is $\left(n-1\right)!$

If the circle can be turned over, then number of circular permutations are $\frac{(n-1)!}{2}$.

# Combination /SELECTION(Order doesnot matter)

## N choose k

Number of ways to choose k elements from a set of n of size n, where order is not important is

$$C\_{k}^{n}=\left(\genfrac{}{}{0pt}{}{n}{k}\right)= \frac{P\_{r}^{n}}{k!}=\frac{n!}{k!\left(n-k\right)!} $$

$\left(\genfrac{}{}{0pt}{}{n}{k}\right)$ is called n choose k. The number of ways to rearrange k elements within themselves is given by k!. Because the order of such rearrangement is not important, such combinations need to be eliminated. This is accomplished by dividing nPk with k!

Notes:

$$\left(\genfrac{}{}{0pt}{}{n}{k}\right)= \left(\genfrac{}{}{0pt}{}{n}{n-k}\right)$$

$$\left(\genfrac{}{}{0pt}{}{n}{0}\right)=1$$

$$\left(\genfrac{}{}{0pt}{}{n}{k+1}\right)+\left(\genfrac{}{}{0pt}{}{n}{k+1}\right)=\left(\genfrac{}{}{0pt}{}{n+1}{k+1}\right)$$

## Pascals triangle

1

1

1

2

1

1

1

1

3

3

1

4

1

6

4

1

1

5

10

10

5

1

1

6

6

15

20

15

Pascals triangle is a geometric arrangement where each row is represented by $\left(\genfrac{}{}{0pt}{}{n}{k}\right)$. Where n corresponds to the row number and k to each element in the row.

This way Pascal’s triangle goes on.

## Sum of all possible combinations

$$C\_{0}^{n}+C\_{1}^{n}+C\_{2}^{n}+…+C\_{n}^{n}=2^{n}$$

The easiest way to prove this, to think of a n digit binary number. Each bit represents a element. If the bit is set means the corresponding element is part of the set. Thus using 2n, all possible combinations of subsets of the set of n elements can be represented. If the selection need to have at least one element, then the total number $2^{n}-1$.

## Sum of all possible combinations if each selection has at least m elements

$$=Sum of all selections – Sum of all way to select (m-1) elements$$

$$=2^{n}- \left[\left(\genfrac{}{}{0pt}{}{n}{0}\right)+\left(\genfrac{}{}{0pt}{}{n}{1}\right)+…+\left(\genfrac{}{}{0pt}{}{n}{m-1}\right)\right]$$

## SUM OF ALL COMBINATIONS if m elements are ALIKE

Out of n elements, m are same, how many selections are possible if we have to select at least 1 element. In this case we assume the resulting selection can have zero or more elements from m.

Number of ways to select m elements$=(m+1)$ (Because all are like, you can select 0 or 1 or 2 or … m)

Number of ways to select zero or more remaining n-m elements$2^{(n-m)}$

Total number of ways is $2^{(n-m)}×(m+1)$ (This includes zero element selection choice as well)

Since we have to select at least one element, we get $2^{(n-m)}×\left(m+1\right)-1$

## Distributing K like elements into n groups

This is also called as "bars and stars" problems as can be shown in the following proof. The rounds represent the elements of the multiset, so there should be k rounds. Number of bars should be equal to n-1. It basically divides the rounds into n groups. Number of rounds in each group correspond to number of elements distributed to that group. Let us take a simple example of distributing 6 elements into 3 groups.

= {Group1 has 2, Group 2 has 1 and Group 3 has 3}

={Group1 has 6, Group 2 has 0 and Group 3 has 0}

Thus the problem of distribution is now transformed into the bar arrangement problem. (i.e.) The number of ways the bars can be arranged. Total number of positions (rounds + bars), that a bar can be placed = n + k -1. In the remining spots place the circles. Hence number of ways the bars can be placed is the same as number of ways to choose n-1 elements from n+k-1.

$$\left(\genfrac{}{}{0pt}{}{k+n-1}{n-1}\right)$$

For the above example of distributing 6 elements into 3 groups is $\left(\genfrac{}{}{0pt}{}{3+6-1}{3-1}\right)=\left(\genfrac{}{}{0pt}{}{8}{2}\right)=\frac{8×7}{2!}=28$ .

Alternatively the same problem can be expressed as: There are 3 types of flowers that you can use in a bouquet. Each bouquet has 6 flowers. Total number of bouquets possible is given by $\left(\genfrac{}{}{0pt}{}{3+6-1}{3-1}\right)$. (i.e.) Number of multisets of size k that can be formed using a set of size n is given by $\left(\genfrac{}{}{0pt}{}{n+k-1}{n-1}\right)$. This is because each group of rounds in the above star and bar representation corresponds to multiplicity of the element. The multisets for the above pictures is

={aabccc}

={aaaaaa}

## Distributing K like elements into n groups with minimum allocation

Number of ways to allocate k like elements into n groups, such that each group should get at least m elements. The solution to this is to first allocate m elements to each group. This will reduce the number of elements available to be distributed to ($k-m×n$). Number of ways ($k-m×n$) can be distributed to k groups is given by

$$\left(\genfrac{}{}{0pt}{}{n+k-m×n-1}{n-1}\right)$$